S.-T. Yau College Student Mathematics Contests 2022
Algebra and Number Theory
Individual

Problem 1. Let M denote the $\mathbb{C}$-vector space consisting of $n \times n$ $\mathbb{C}$-matrices. Let Id denote the identity matrix.

Please determine all $\mathbb{C}$-linear functions σ on M such that $\sigma(AB) = \sigma(BA)$ and $\sigma(\mathrm{Id}) = n$.

Problem 2. Let p be a prime number and F a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers.

(a) Suppose $p \neq 2$. Prove that every element of $\mathcal{O}_F$ can be written as a sum of three squares in $\mathcal{O}_F$.

(b) Suppose $F = \mathbb{Q}_2$. Prove that every element of $\mathcal{O}_F$ can be written as a sum of four squares in $\mathcal{O}_F$.

Problem 3. Let $R = \prod_p \mathbb{F}_p$, where p runs through all prime numbers.

(a) Show that there exists a maximal ideal $\mathfrak{m}$ of R such that $R/\mathfrak{m}$ is a field of characteristic zero in which -1 is not a square.

(b) Show that there is no maximal ideal $\mathfrak{n}$ of R admitting an embedding $R/\mathfrak{n} \hookrightarrow \mathbb{R}$.